*The triangulation is based on Delaunay method, creating the corresponding Voronoi polygons and is tested to process 1 million of points in less than 10 seconds (din referat) -> continuare*

## Triangulation definition

The meaning of triangulation is to generate a mesh of triangles from a given set of points. The triangles are formed by edges between the points of the set.

Given two points Pi and Pj in the plane T, the perpendicular to the segment PiPj in the middle point divides the plane T into two regions, Vi and Vj. Region Vi contains all and only the points closest to Pi than to Pj ; if we have more points we can easily extend this concept saying that Vi is the region assigned to Pi so that each point belonging to Vi is closest to Pi than to any other point.

The subdivision of the space determined by a set of distinct points so that each point has associated with it the region of the space nearer to that point than to any other is called **Dirichlet tessellation**.

This process applied to a closed domain generates a set of convex distinct polygons called Voronoi regions which cover the entire domain. This definition can be extended to higher dimension where, for example in three dimensions, the Voronoi regions are convex polyhedrons. If we connect all the pairs of points sharing a border of a Voronoi region we obtain a triangulation of the convex space containing those points. This triangulation is known as **Delaunay triangulation**. An example of the relationship between Voronoi regions and Delaunay triangulation in two dimensions is given in fig. 1. Similarly we can obtain a triangulation for higher dimensions, for example in three dimensions if we connect all pairs of points sharing a common facet in the Voronoi diagram, the result is a set of tetrahedra filling the entire domain.

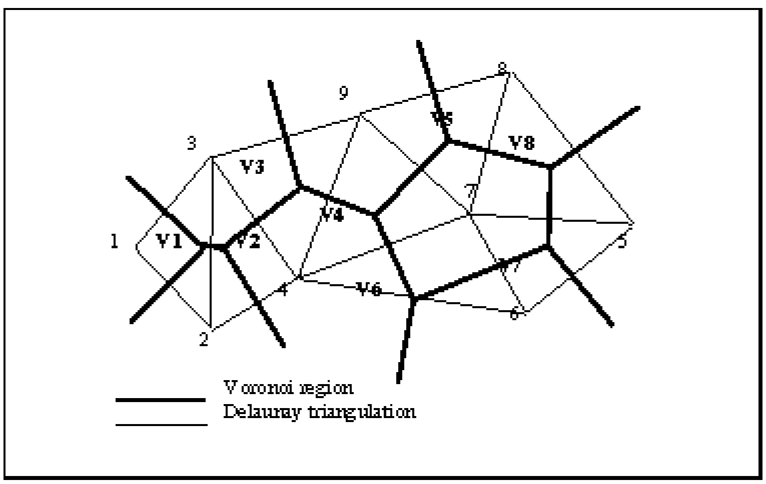


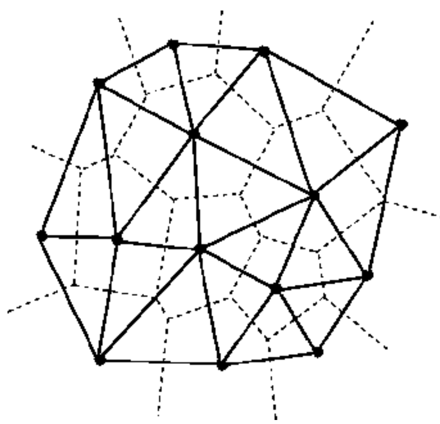

Fig. 1 - Voronoi regions and associated Delaunay triangulation

The Delaunay triangulation is the dual structure of the Voronoi diagram in $R^2$. By dual, we mean to draw a line segment between two Voronoi vertices if their Voronoi polygons have a common edge, or in more mathematical terminology: there is a natural bijection between the two which reverses the face inclusions.
A Voronoi diagram (with respect to the Euclidian metric d: $R^2 \times R^2 \rightarrow R$) of a set

$\Omega = \{g_1, \ldots g_n\} \subset R^2$ of *n* generators $g_1 \ldots g_n$ is the collection of *n* poligons

$$P_i = \{p \in R^2 | d(p, g_i) \leq d(p, g_j) \forall_j \in \{1, \ldots n\}\}$$

The “skeleton” formed by the boundaries of these polygons can be viewed as a planar graph, whose dual graph is called Delaunay triangulation of Ω. Because such a Delaunay triangulation contains much information about the neighborhood structure of the generators it is used in several algorithms



Delaunay triangulation, on top of the Voronoi diagram (in dotted lines)

For a set P of points in the (*d*-dimensional) Euclidean space, a Delaunay triangulation is a triangulation DT(P) such that no point in P is inside the circum-hypersphere of any simplex in DT(P).

It is known that there exists a unique Delaunay triangulation for P, if P is a set of points in *general position*; that is, there does not exists a *k*-flat containing $k + 2$ points nor a *k*-sphere containing $k + 3$ points, for $1 \leq k \leq d - 1$ (e.g., for a set of points in $R^3$; no three points are on a line, no four on a plane, no four are on a circle, and no five on a sphere).

The problem of finding the Delaunay triangulation of a set of points in *n*-dimensional Euclidean space can be converted to the problem of finding the convex hull of a set of points in $(n + 1)$-dimensional space, by giving each point $p$ an extra coordinate equal to $|p|^2$, taking the bottom side of the convex hull, and mapping back to *n*-dimensional space by deleting the last coordinate. As the convex hull is unique, so is the triangulation, assuming all facets of the convex hull are simplices. A facet not being a simplex implies that $n + 2$ of the original points lay on the same $d$- hypersphere, and the points were not in general position.

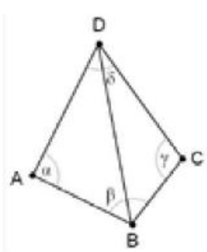
# Properties

Let $n$ be the number of points and $d$ the number of dimensions.

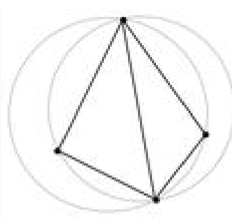
- The union of all simplices in the triangulation is the convex hull of the points.
- The Delaunay triangulation contains at most $O(n^{\lceil d/2 \rceil})$ simplices.
- In the plane ($d = 2$), if there are $b$ vertices on the convex hull, then any triangulation of the points has at most $2n - 2 - b$ triangles, plus one exterior face
- In the plane, each vertex has on average six surrounding triangles.
- In the plane, the Delaunay triangulation maximizes the minimum angle. Compared to any other triangulation of the points, the smallest angle in the Delaunay triangulation is at least as large as the smallest angle in any other. However, the Delaunay triangulation does not necessarily minimize the maximum angle.
- A circle circumscribing any Delaunay triangle does not contain any other input points in its interior.
- If a circle passing through two of the input points doesn't contain any other of them in its interior, then the segment connecting the two points is an edge of a Delaunay triangulation of the given points.
- The Delaunay triangulation of a set of points in $d$-dimensional spaces is the projection of the points of convex hull onto a ($d$+1)-dimensional paraboloid.
- The closest neighbor $b$ to any point $p$ is an edge $bp$ in the Delaunay triangulation since the nearest neighbor graph is a subgraph of the Delaunay triangulation.

From the above properties an important feature arises: Looking at two triangles ABD and BCD with the common edge BD (see figures), if the sum of the angles α and γ is less than or equal to 180°, the triangles meet the Delaunay condition.

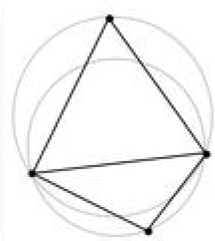
This is an important property because it allows the use of a *flipping* technique. If two triangles do not meet the Delaunay condition, switching the common edge BD for the common edge AC produces two triangles that do meet the Delaunay condition:




This triangulation does not meet the Delaunay condition (the sum of α and γ is bigger than 180°).



This triangulation does not meet the Delaunay condition (the circumcircles contain more than 3 points).



*Flipping* the common edge produces a Delaunay triangulation for the four points.

Many algorithms for computing Delaunay triangulations rely on fast operations for detecting when a point is within a triangle's circumcircle and an efficient data structure for storing triangles and edges. In two dimensions, one way to detect if point *D* lies in the circumcircle of *A*, *B*, *C* is to evaluate the determinant:

$$\begin{vmatrix} A_x & A_y & A_x^2 + A_y^2 & 1 \\ B_x & B_y & B_x^2 + B_y^2 & 1 \\ C_x & C_y & C_x^2 + C_y^2 & 1 \\ D_x & D_y & D_x^2 + D_y^2 & 1 \end{vmatrix} = \begin{vmatrix} A_x - D_x & A_y - D_y & (A_x^2 - D_x^2) + (A_y^2 - D_y^2) \\ B_x - D_x & B_y - D_y & (B_x^2 - D_x^2) + (B_y^2 - D_y^2) \\ C_x - D_x & C_y - D_y & (C_x^2 - D_x^2) + (C_y^2 - D_y^2) \end{vmatrix} > 0$$

Assuming *A*, *B* and *C* to lie counter-clockwise, this is positive if and only if *D* lies in the circumcircle.

For modeling terrain or other objects given a set of sample points, the Delaunay triangulation gives a nice set of triangles to use as polygons in the model. In particular, the Delaunay triangulation avoids narrow triangles (as they have large circumcircles compared to their area).